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RECENT RESULTS ON CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS: A UNIFIED APPROACH THROUGH EXTENSIONS OF DENY'S THEOREM

ABSTRACT

The problem of identifying solutions of general convolution equations relative to a group has been studied in two classical papers by Choquet and Deny (1960) and Deny (1961). Recently, Lau and Rao (1982) have considered the analogous problem relative to a certain semigroup of the real line, which extends the results of Marsaglia and Tubilla (1975) and a lemma of Shanbhag (1977). The extended versions of Deny's theorem contained in the papers by Lau and Rao, and Shanbhag (which we refer to as LRS theorems) yield as special cases improved versions of several characterizations of exponential, Weibull, stable, Pareto, geometric, Poisson and negative binomial distributions obtained by various authors during the last few years. In this paper we review some of the recent contributions to characterization of probability distributions (whose authors do not seem to be aware of LRS theorems or special cases existing earlier) and show how improved versions of these results follow as immediate corollaries to LRS theorems. We also give a short proof of Lau-Rao theorem based on Deny's theorem and thus establish a direct link between the results of Deny (1961) and those of Lau and Rao (1982). A variant of Lau-Rao theorem is proved and applied to some characterization problems.

Key Words: Characterizations, Deny's theorem, Exchangeable random variables, Exponential, geometric, Pareto and stable distributions, Integrated Cauchy functional equations, Lau-Rao theorem, Shanbhag's lemma.

1. INTRODUCTION

Let S be such that it equals either R $(=(-\infty,\infty))$ or R_+ $(=[0,\infty))$, σ be a measure on (the Borel σ-field of) S such that $(\{0\}^c) > 0$, and $H : S \longrightarrow R_+$ be a non-negative continuous function which satisfies the functional equation

$$H(x) = \int_S H(x + y)\sigma(dy),\ \forall\ x \in S. \tag{1.1}$$

From a general theorem of Deny (1961), it follows that if S = R, then either $H(x) \equiv 0$ or

$$H(x) = \xi_1(x)\exp(-\eta_1 x) + \xi_2(x)\exp(-\eta_2 x),\ x \in S \tag{1.2}$$

with η_i such that

$$\int_S e^{-\eta_i x}\sigma(dx) = 1,\ i = 1, 2$$

and ξ_i as non-negative periodic functions such that

$$\xi_i(x + y) = \xi_i(x),\ \forall\ x \in S \text{ and } y \in \text{supp } \sigma,$$

for i = 1, 2. (Observe that if S = R and $H \equiv 0$, then the measure σ involved in (1.1), has to be a Radon measure). As a corollary of Deny's general theorem, we have Choquet and Deny (1960) theorem which has important applications in renewal theory (Feller, 1966 Vol. 2, p. 351).

Recently, Lau and Rao (1982) solved the equation (1.1) when S = R_+ which subsumes partial results given by Marsaglia and Tubilla (1975), Klebanov (1977), Shanbhag (1977), Shimuzu (1978) and Ramachandran (1979). A simpler

proof of Lau-Rao theorem appears in Ramachandran (1982). More recently, Alzaid, Rao and Shanbhag (1983) used an argument based on de Finneti's theorem concerning exchangeable random variables to derive the same result. Davies and Shanbhag (1984) have given a martingale proof for an extended version of Deny's result which generalizes Lau-Rao theorem. Extensions of Deny's general theorem to the case of a semigroup have also been given via other approaches by Richards (1981) and Ressel (1984) among others. However, both Richards and Ressel were able to deal with the problem only under some stringent conditions which imply that the semigroup generated by the support of the measure in the functional equation equals the semigroup itself; additionally Richards requires the function to be bounded and Ressel requires the semigroup to be countable.

Various applications of Lau and Rao (1982) theorem and Shanbhag's (1977) lemma, which we refer to as LRS (Lau-Rao-Shanbhag) theorems have been considered by Lau and Rao (1982), M. B. Rao and Shanbhag (1982), Rao (1983), Alzaid (1983) and Davies and Shanbhag (1984) with special reference to damage models, order statistics, record values, lack of memory, reliability and renewal theories. The main purpose of this paper is to indicate further applications of LRS theorems by reviewing some of the recent contributions to characterizations of probability distributions, e.g., Dallas (1981), Deheuvels (1984), Gupta (1984) and Grosswald, Kotz and Johnson (1980), whose authors do not seem to be aware of LRS theorems or the special cases existing earlier. We show that LRS theorems not only provide a unified approach to a wide variety of characterizations of distributions such as Poisson, Pareto, Weibull, stable, geometric and negative binomial, but their application leads in many cases to improved versions of the results already available in the

literature. In this paper we also give a simple proof of Lau-Rao theorem via Deny's theorem and thus obtain a direct link between the two theorems. In addition, we investigate the problem of solving the integral equation

$$(1.3) \qquad \alpha + \beta f(x) = \int_{R_+} f(x+y)\mu(dy) \text{ a.e.}[L] \ \forall\, x \in R_+$$

where α and β are constants and indicate its applications to characterization problems.

2. LAU-RAO THEOREM

Lau-Rao theorem. Let H be a non-negative real locally integrable measurable function on R_+, which is not a function identically equal to zero a.e. [L], (L indicating Lebesgue measure), satisfying the functional equation

$$(2.1) \qquad H(x) = \int_{R_+} H(x+y)\sigma(dy) \text{ a.e.}[L] \text{ for } x \in R_+$$

where σ is a σ-finite measure on R_+ such that $\sigma(\{0\}) < 1$. Then one of the two possibilities hold.

(a) σ in (2.1) is arithmetic with some span λ and

$$H(x + n\lambda) = H(x)b^n, \ n = 0, 1, \ldots, \text{ a.e.}[L] \text{ for } x \in R_+$$

with b such that

$$\sum_{n=0}^{\infty} b^n \sigma(\{n\lambda\}) = 1.$$

(b) σ in (2.1) is non-arithmetic and

$$H(x) \propto \exp\{-\eta x\} \text{ a.e.}[L] \text{ for } x \in R_+$$

with η such that

$$\int_{R_+} \exp\{-\eta x\}\sigma(dx) = 1.$$

Lau and Rao (1982) gave a self-contained real variable proof of the above theorem. We now present an alternative proof based on Deny's (1961) theorem, which provides an important link and at the same time brings out the main difference between the two theorems.

<u>Proof</u> There is no loss of generality in assuming that $\sigma(R_+) > 1$. Consider some $d > 0$, and define

$$(2.2) \qquad \hat{H}(x) = \int_0^d H(x + y)dy, \quad x \in R_+.$$

Clearly $\hat{H}$ is continuous and in view of Fubini's theorem satisfies (2.1) with the statement 'a.e.[L]' deleted. From elementary Lemma 1 of Davies and Shanbhag (1984), it immediately follows that for every support point s of σ

$$(2.3) \qquad \hat{H}(x + 2s)\hat{H}(x) \geq [\hat{H}(x + s)]^2, \quad x \in R_+.$$

We can choose d sufficiently large so that $\hat{H}(0) > 0$ and hence $\hat{H}(s_0) > 0$ for some positive support point s_0 of σ. From (2.3), it follows that $\hat{H}(2s_0) > 0$. Consequently, for sufficiently large d, we have $\hat{H}(x) > 0 \ \forall \ x \in [0, 2s_0]$ for some positive support point s_0 of σ. We shall now fix the s_0 in question. From (2.3), we can then claim that $\hat{H}(x) > 0 \ \forall \ x \in R_+$ and

$$\left\{ \frac{\hat{H}(x + ns_0)}{\hat{H}(x + \overline{n - 1}s_0)} : n = 1, 2, \ldots \right\}$$

is an increasing sequence for each $x \in R_+$. Clearly then we have

$$(2.4) \qquad \hat{H}(x + s_0) \geq v\, \hat{H}(x), \quad x \in R_+,$$

where

$$v = \inf \left\{ \frac{\hat{H}(x + s_0)}{\hat{H}(x)} : 0 \leq x \leq s_0 \right\} > 0.$$

There is no loss of generality in assuming $\sigma(\{0\}) = 0$. If σ is arithmetic, or more generally if there exists a constant $c > 0$ such that $\sigma((0,c)) = 0$ (i.e., if 0 is not a cluster point of the support of σ), define $\hat{\sigma} = \sigma$. Otherwise,

considering c such that $\sigma((0,c)) < 1$ and $0 < c < s_0$, define

$$\hat{\sigma}(\cdot) = \sum_{n=1}^{\infty} \sigma_n([c,\infty)\cap\cdot)$$

to be a measaure on R_+, where $\sigma_1 = \sigma$, and for each $n \geq 2$, σ_n is the convolution of the measures $\sigma_{n-1}([0,c)\cap\cdot)$ and σ. It is then obvious that $\hat{\sigma}([0,c)) = 0$ and

$$(2.5) \qquad \hat{H}(x) = \int_{R_+} \hat{H}(x + y)\hat{\sigma}(dy), \; x \in R_+.$$

Observe that the measure $\hat{\sigma}$ defined here is such that it is arithmetic if σ is arithmetic, and in that case, both have the same support; also $\hat{\sigma}$ is non-arithmetic if σ is non-arithmetic. Define now inductively $\hat{H}(x)$ for $x \in [-(n + 1)c, -nc)$ for $n = 0, 1, 2,\ldots$ such that

$$(2.6) \qquad \hat{H}(x) = \int_{[c,\infty)} \hat{H}(x + y)\hat{\sigma}(dy).$$

It is then easily seen, especially in view of (2.4), (2.5), and (2.6), that we have a continuous function $\hat{H} : R \longrightarrow R_+$ such that its restriction to R_+ agrees with our original $\hat{H}$ and

$$(2.7) \qquad \hat{H}(x) = \int_{R_+} \hat{H}(x + y)\hat{\sigma}(dy), \; x \in R.$$

From Deny's (1961) theorem, it then follows that

$$\hat{H}(x) = \xi(x)e^{-\eta x}, \; x \in R$$

for some $\eta > 0$ and some function ξ satisfying the condition $\xi(x + s) = \xi(x)$, $x \in R$ for each support point s of $\hat{\sigma}$. The required result now follows on noting that

$$\int_x^\infty H(y)dy = \frac{\eta\xi(0)}{1 - \exp(-\eta d)} \int_x^\infty e^{-\eta y}dy \quad \forall\, x \geq 0$$

if σ is non-arithmetic, and

$$\infty > \int_x^\infty H(y + n\lambda)dy = \int_x^\infty H(y)\exp(-n\lambda\eta)dy \quad \forall\, x \geq 0,\ n = 0, 1, \ldots$$

if σ is arithmetic with span λ and

$$\int_{R_+} \exp(-\eta x)\sigma(dx) = 1.$$

Remark 1 If the conditions in Lau-Rao theorem are met with R_+ replaced by R, then it follows at once from Deny's theorem that

$$(2.8) \qquad n\int_0^{1/n} H(x + y)dy = \xi_1^{(n)}(x)\exp(-\eta_1 x) + \xi_2^{(n)}(x)\exp(-\eta_2 x)$$

for $n = 1, 2, \ldots$, a.e.[L] for $x \in R$, where η_1 and η_2 are as defined in (1.2) and $\xi_1^{(n)}$ and $\xi_2^{(n)}$ are of the form of ξ_1 and ξ_2 in (1.2) with $S = R$. Since H is locally integrable, it follows that

$$\lim_{n\to\infty} n\int_0^{1/n} H(x + y)dy = H(x), \text{ a.e.[L] for } x \in R$$

and hence that if s_0 is any nonzero support point of σ (which clearly exists), then

$$(2.9) \quad n\int_0^{1/n} H(x+y)dy \longrightarrow H(x) \text{ and } n\int_0^{1/n} H(x+s_0+y)dy \longrightarrow H(x+s_0)$$

as $n \to \infty$, a.e.[L] for $x \in R$.

Consequently, in view of (2.8), it follows that there exist functions ξ_1 and ξ_2 as in (1.2) such that $\xi_1^{(n)} \to \xi_1$ and $\xi_2^{(n)} \to \xi_2$ as $n \to \infty$, a.e.[L] on R, and hence such that

$$(2.10) \quad H(x) = \xi_1(x)\exp(-\eta_1 x) + \xi_2(x)\exp(-\eta_2 x) \text{ a.e.[L] for } x \in R.$$

The result (2.10) was established by Lau and Rao (1984a).

Incidentally, it may be noted here that in the case of nonarithmetic σ, the form of H in (2.10) simplifies to

$$(2.11) \quad H(x) = \beta\exp(-\eta_1 x) + (1-\beta)\exp(-\eta_2 x) \text{ a.e.[L] for } x \in R,$$

where β is some constant in [0,1].

<u>Remark 2</u> From Davies and Shanbhag (1984), it is evident that at least in the case of continuous H, Lau-Rao theorem remains valid even when the requirement of σ-finiteness of the measure σ is dropped. (This is so more clearly in the case of Remark 1.) However, the following example shows that the general result of Lau-Rao theorem does not remain valid if σ is not σ-finite.

Example 1 Let σ be that measure on R_+ for which its restriction to (1,2] agrees with the counting measure on (1,2] and $\sigma([0,1] \cup (2,\infty)) = 0$. Define a function $H : R_+ \to R_+$ such that

$$H(x) = \begin{cases} 1 \text{ if } x \in [0,1) \text{ and } x = 2, \\ 0 \text{ otherwise.} \end{cases}$$

Observe that we have here

$$H(x) = \int_{R_+} H(x + y)\sigma(dy) \text{ a.e. } [L] \text{ for } x \in R_+$$

but H is not of the form as in Lau-Rao theorem.

Remark 3 As observed by Lau and Rao (1982) and Alzaid, Rao and Shanbhag (1983), Lau-Rao theorem yields the following modified version of Shanbhag's (1977) lemma.

Lemma Let $\{(v_n, w_n) : n = 0, 1, \ldots\}$ be a sequence of vectors with non-negative real components such that $v_n \neq 0$ for at least one n, $w_0 < 1$, and the largest common divisor of the set $\{n : w_n > 0\}$ is unity. Then

$$v_m = \sum_{n=0}^{\infty} v_{m+n} w_n, \quad m = 0, 1, \ldots$$

if and only if

$$v_n = v_0 b^n, \quad n = 0, 1, 2, \ldots \text{ and } \sum_{n=0}^{\infty} w_n b^n = 1$$

for some $b > 0$.

The modified version of Shanbhag's lemma yields somewhat improved versions of the general characterization theorems for the univariate and

bivariate cases given in Shanbhag (1977) as discussed in Shanbhag (1983). Further variants and extensions of Shanbhag's (1977) lemma have been considered, among others, by Alzaid, Rao and Shanbhag (1983) and Lau and Rao (1984b).

3. COMMENTS ON RECENT RESULTS

In this section we review some recent contributions to characterization of probability distributions, comment on the gaps in the proofs and show how improved versions of the results can be obtained by using LRS theorems.

3.1 Gupta (1984)

One of the main theorems (Theorem 3.1) of Gupta (1984) is that "$E[(R_{j+1} - R_j)^r | R_j = y] = c$ (independent of y) for fixed j and $r \geq 1$ iff F is exponential, where $R_1, R_2, \ldots$ are record values from a continuous distribution function F such that $F(0) = 0$." We have the following comments on the statement and proof of Gupta's theorem.

Gupta mentions that the condition on conditional expectation in his theorem implies that

$$(3.1.1) \qquad c = \int_0^\infty r u^{r-1} \frac{S(u+y)}{S(y)} \, du$$

where $S(x) = 1 - F(x)$, or

$$(3.1.2) \qquad c\, S(y) = \int_0^\infty r u^{r-1} S(u+y) du.$$

But for (3.1.2) to be valid for all $y \in (0,\infty)$, it is necessary to assume, besides continuity of F, that $F(x) > 0$ for $x > 0$, which is not explicitely mentioned in the statement of the theorem. Once (3.1.2) is assumed to be valid for all y, then an application of Lau-Rao theorem immediately shows that $S(x) = e^{-\lambda x}$ which is the required result.

However, Gupta obtains the solution in a different way by considering Mellin's transform of both sides of (3.1.2), deriving an equation of the form

$$(3.1.3) \qquad h(t) - Ah(t - r) = 0 \text{ for } t > r,$$

and writing its solution as $h(t) = ke^{bt}$ attributed to Bellman and Cooke (1963, p. 54). Unfortunately (3.1.3) has no unique solution; for instance,

$$h(t) = \exp(\text{[illegible]} \ \lambda \sin \frac{2\pi t}{r}) \quad , \lambda \neq 0$$

is also a solution, which shows that further arguments are necessary to justify Gupta's solution.

The same remark applies to the alternative proof given by Srivastava and Singh (1975, p. 273) for the Rao-Rubin (1964) theorem, quoting the Bellman-Cooke result.

As observed above, the statement of Gupta's theorem needs the additional condition, $0 < F(x)$ for $x > 0$. Some extensions of Gupta's result are as follows:

(i) The result is true even if $0 < r < 1$ since Lau-Rao theorem is still applicable.

(ii) If F is such that $F(a) = 0$ and $F(x) > 0$ for $x > a$, then the characterization is valid but for a modification of F as exponential with a shift in the origin.

(iii) Lau-Rao theorem also implies that the same characterization is obtained if in Gupta's condition, the expression $(R_{j+1} - R_j)^r$ is replaced by $\phi(R_{j+1} - R_j)$ where ϕ is an increasing or decreasing real function on R_+ with $\phi(x) \neq \phi(0+)$ $\forall$ $x > 0$ and such that $E[|\phi(R_{j+1} - R_j)|] < \infty$. As a special case of this result, it follows that

$$E\{[1 - \exp(R_{j+1} - R_j)] | R_j\} = \text{constant a.e.}$$

characterizes an exponential distribution (but for a shift).

(iv) If F is arithmetic with its support as $\{0, 1, 2, \ldots\}$, then the condition $E[(R_{j+1} - R_j)^r | R_j = y] = c$ (independent of y) implies that

$$F(s) - F(s-) \begin{cases} = \text{arbitrary for } s = 0, \ldots, j-1 \\ \propto \ b^s \text{ for } s = j, j+1, \ldots \, . \end{cases}$$

The result is obtained by an application of Shanbhag's lemma. The last result remains valid even when the expression $(R_{j+1} - R_j)^r$ is replaced by $\phi(R_{j+1} - R_j)$ where $\{\phi(n): n = 0, 1, \ldots\}$ is an increasing or decreasing real sequence with $\phi(0) \neq \phi(1) \neq \ldots \neq \phi(j)$ and such that $E\{|\phi(R_{j+1} - R_j|\} < \infty$.

3.2 Grosswald, Kotz and Johnson (1980)

Grosswald et al (1980) proved that if F_2 is a distribution function on R_+ with survivor function S_2 satisfying $S_2(0) = 1$ and having a power series expansion, then

$$(3.2.1) \quad \int_{[0,t]} S_2(t - x) F_1(dx) = \int_{[0,t]} [S_2(t)/S_2(x)] F_1(dx), \ t \in R_+$$

for every distribution function F_1 on R_+ with $F_1(0) = 0$ if and only if F_2 is exponential. (In (3.2.1), $S_2(t)/S_2(x)$ is interpreted as zero if $S_2(x) = 0$). They conjectured that the result is still true when S_2 (or equivalently F_2) is merely assumed to be continuous. More recently, Westcott (1981) used a probabilistic argument to show that the conjecture is correct.

However, an improved version of the above result follows trivially from the result of Marsaglia and Tubilla (1975): Let F_2 be a probability distribution on R_+ such that $S_2(0) > 0$ and x_1, x_2 be two positive numbers such that $S_2(x_2) > 0$, $x_1 < x_2$ and x_1/x_2 is irrational. If (3.2.1) is satisfied for

any two distinct probability distributions F_1 concentrated on $\{x_1, x_2\}$, then F_2 is exponential. (If F_2 is exponential, then (3.2.1) is satisfied for any F_1 on R_+.)

It is interesting to point out that this result does not hold when the condition 'any two distinct probability distributions' is replaced by 'a probability distribution.' This is illustrated by the following example.

Example 2 Let $x_2 < 2x_1$ and F_1 be a probability distribution on R_+ such that $F_1(x_1) - F_1(x_1-) = \alpha$ and $F_1(x_2) - F_1(x_2-) = 1 - \alpha$ where $0 \leq \alpha \leq 1$. Define a probability distribution function F_2 on R_+ such that

$$F_2(x) = \begin{cases} 0 \text{ if } x < x_1, \\ \beta \text{ if } x_1 \leq x \leq x_2,\ 0 < \beta < 1, \\ \beta + (1 - \beta)\{\alpha F_2(x - x_1) + (1 - \alpha)F_2(x - x_2)\} \text{ if } x > x_2. \end{cases}$$

Clearly F_2 is a distribution function and satisfies (3.2.1).

It is possible to give several other variants of our modified version of the result of Grosswald et al (1980) such as the arithmetic analogue characterizing the geometric distribution.

3.3 Dallas (1981)

In this paper, it is shown that if $R_0, R_1, \ldots$ are record values from a continuous distribution function $F(x)$ and $0 \leq i < j$ are fixed integers, then the independence of $R_j - R_i$ and R_i implies that F is exponential or shifted exponential.

There is an implicit assumption in the proof given by Dallas that every point of $[a, \infty)$ is a support point of F. Further, the derivation of the

conditional distribution (Dallas, 1981, p. 950) needs some justification especially if one is dealing with a distribution having a singular continuous component in its Lebesgue decomposition. We provide the sketch of an alternative and more satisfactory proof based on LRS theorems.

Let for any $c < b$, the right extremity of the distribution function F, $\{R_n^{(c)}, n = 1, 2, \ldots\}$ be a sequence of record values from the distribution

$$F_c(x) = \begin{cases} \dfrac{F(x) - F(c)}{1 - F(c)} & \text{if } x > c, \\ 0 & \text{otherwise.} \end{cases}$$

It is easily seen that $R_j - R_i$ is independent of R_i, if and only if the distribution of $R_{j-i}^{(c)} - c$ is independent of c a.e.[F]. The distribution of $R_{j-i}^{(c)} - c$ is computed to be

$$(3.3.1) \quad P\{R_{j-i}^{(c)} - c \le x\} = \begin{cases} [(j-i-1)!]^{-1}\int_\alpha^1 (-\log y)^{j-i-1}dy & \text{if } c \le c + x < b \\ 1 & \text{if } c + x \ge b \\ 0 & \text{if } x < 0, \end{cases}$$

where $\alpha = [1 - F(c+x)]/[1 - F(c)]$. (A rigorous proof of (3.3.1) follows from a lemma in Kotz and Shanbhag, 1980.) Consequently, it follows that $R_j - R_i$ is independent of R_i if and only if $b = \infty$ and $[1 - F(c + x)]/[1 - F(c)]$ is independent of c a.e.[F]. Then from the result of Marsaglia and Tubilla (1975) (and not from the usual lack of memory property of an exponential distribution as mentioned by Dallas), it then follows that F is either exponential or shifted exponential. (Observe that if $[1 - F(c + x)]/[1 - F(c)]$, $x \ge 0$ is independent of c a.e.[F], then the left extremity of F should be some $a > -\infty$ and for every $c \in \text{supp}\,[F]$ and $x \in R_+$

$$1 - F(c + x) = [1 - F(x + a)][1 - F(c)] \text{ a.e.[F]}).$$

3.4 Deheuvels (1984)

This paper reviews some of the characterization results on the exponential and geometric distributions based on properties of order statistics and record values. As mentioned in our present discussion and elsewhere, several of these results and improved versions in some cases can be deduced directly from LRS theorems, which provide a unified approach to a wide variety of problems. However, we make a comment on Theorem 2 of Deheuvels (1984) which is, perhaps, only of minor relevance to this discussion. The theorem mentioned is not valid unless in the statement above equation (7) on p. 329 of the paper, 'for all z' is replaced by 'for $z \in R$ a.e.[F]' and $1 - F(x)$ on the right hand side of (7) is changed to $1 - F(z)$. (The latter of the two errors in question appears to be a misprint).

3.5 Rao (1983)

We give here slight refinements and extensions of some of the results mentioned in Rao (1983), which again follow from Lau-Rao theorem.

Theorem 5.1 of Rao (1983) states: Let the distribution function F of a r.v. X be continuous and such that $F(0) = 0 \leq F(x) < 1$ for all $x \in [0,\infty)$. Then $F(x) = 1 - e^{-\lambda x}$ if and only if $R_{j+1} - R_j$ and R_1 have the same distribution, where $R_1, R_2, \ldots,$ are record values. This theorem remains true even if F is assumed to be such that $F(0) = 0$, the right extremity is not a discontinuity point, and at least one of its support points is a continuity point.

Theorem 5.2 of Rao (1983) states: Let X be a discrete r.v. taking values 0, 1,... such that $p_i = P(X = i) > 0$ for all i. Then X has a geometric distribution if $R_{j+1} - R_j$ has the same distribution as $R_1 + 1$. This theorem remains true if instead of $p_i > 0$ for all i, we have only $\sup\{i : p_i > 0\} = \infty$ and $p_i > 0$ for $i = 0,\ldots,j + 1$.

Theorem 4.3 of Rao (1983) states: Let $X_{(1)} \leq X_{(2)}$ be order statistics in a sample of size 2 from a discrete distribution on {0, 1, 2,...} with $P(X = i) = p_i \neq 0$ for all i. Then

$$E(X_{(2)} - X_{(1)} | X_{(1)} = x) = \mu \text{ for } x = 0, 1,\ldots \tag{3.5.1}$$

iff X has a geometric distribution.

The condition (3.5.1) implies that

$$\mu(G_r + G_{r+1}) = 2(G_{r+1} + G_{r+2} + \ldots),\ r = 0, 1,\ldots \tag{3.5.2}$$

where $G_r = p_r + p_{r+1} + \ldots$. Clearly (3.5.2) is equivalent to

$$\mu G_r = 2G_{r+1} + \mu G_{r+2},\ r = 0, 1,\ldots \tag{3.5.3}$$

and the desired result follows from (3.5.3) by applying Shanbhag's lemma. (The expression (4.4) in Rao (1983) should be as in (3.5.2).)

A stronger version of the above result is obtained by replacing the condition (3.5.1) by

$$E(\phi(X_{(2)} - X_{(1)})| X_{(1)} = x) = \mu \text{ for } x = 0, 1, \ldots$$

where ϕ is such that $E[| \phi(X_{(2)} - X_{(1)})|] < \infty]$, $\phi(1) > \phi(0)$ and $\phi(r + 2) - 2\phi(r + 1) + \phi(r) \geq 0$ for all r, i.e., the second differences of ϕ are non-negative.

Another version of Theorem 4.3 is obtained by considering only samples without ties, in which case $X_{(2)} > X_{(1)}$. Let ϕ be an increasing function such that $\phi(2) - \phi(1) > \phi(1)$. Then

$$E(\phi(X_{(2)} - X_{(1)})|X_{(1)} = x) = \mu, \; x = 0, 1, \ldots$$

implies that

$$p_i \propto \beta^i, \; i = 1, 2, \ldots$$

for some $\beta \in (0,1)$ and p_0 is arbitrary. (Slightly stronger results than those discussed here follow via the extended version of Shanbhag's lemma given in section 2; the results are also valid when $-\phi$ meets the requirements of ϕ).

Finally Theorem 6.2 of Rao (1983), in which some assumptions are not explicitly mentioned, can be stated as follows: Let x be a non-negative random variable with a continuous distribution function F, and h be a real function on $[1,\infty)$ such that it is either increasing or decreasing with $h(x) \neq h(1+)$ for each $x > 1$. If

$$E[h\left(\frac{X}{a}\right)|X \geq a] = \text{constant} \; \forall \; a \in (0,\infty)$$

then X has a Pareto distribution. (The result in question remains valid even when the assumption of continuity of F is replaced by $F(0) = 0$; also the extended result remains valid when the assumption that $h(x) \neq h(1+)$ for each $x > 1$ is replaced by that there exist points x_1, $x_2 > 1$ such that $h(x + x_i) \neq h(x_i-)$, $i = 1, 2$ for each $x > 0$ and $\log x_1/\log x_2$ is irrational).

4. A VARIANT OF THE LAU-RAO THEOREM

Consider the following equation which is a variant of the one discussed by Lau and Rao (1982):

$$\int_{R_+} f(x + y)\mu(dy) = f(x) + c \quad \text{a.e.}[L] \text{ for } x \in R_+ \tag{4.1}$$

where $f : R_+ \to R$ is a locally integrable Borel measurable function and μ is a σ-finite measure on R_+ with $\mu(\{0\}) < 1$. (The identity in (4.1) is understood as the one for which the left hand side exists and equals the right hand side.) This may clearly be viewed as an integrated version of the equation $f(x + y) = f(x) + f(y)$ which is derived from the Cauchy equation by taking logarithms. In this case, by analogy with Lau-Rao theorem, one would be tempted to conjecture that a solution of (4.1) is a.e. of the same form as the logarithm of a positive solution of Lau-Rao equation. However, we have the following counter example to show that such a conjecture cannot hold.

<u>Example 3</u> Consider μ to be a probability measure on R_+ such that it is determined by an infinitely divisible probability distribution with an entire characteristic function. From Picard's theorem (c.f. Titchmarsh, 1949, p. 277) and the fact that the characteristic function involved here does not vanish, we can conclude that there exist infinitely many points (a_r, b_r) of R^2 such that

$$\int_{R_+} e^{a_r x + i b_r x} \mu(dx) = 1$$

or equivalently such that

$$\int_{R_+} e^{a_r x} \cos(b_r x)\mu(dx) = 1 \tag{4.2}$$

and

$$(4.3) \qquad \int_{R_+} e^{a_r x} \sin(b_r x)\mu(dx) = 0.$$

If we now define

$$f_r(x) = e^{a_r x} \cos(b_r x), \; x \in R_+,$$

it follows immediately, in view of (4.2) and (4.3) that

$$\int_{R_+} f_r(x + y)\mu(dy) = f_r(x), \; x \in R_+$$

which shows that the conjecture cannot be true.

If we replace R_+ by R in the problem considered, we arrive at the variant of Lau-Rao problem mentioned in Remark 1. In this latter case, we have a simpler counter example on taking $f(x) = x^2$ and μ as any probability distribution with zero mean and finite variance. Clearly we have then

$$\int_R f(x + y)\mu(dy) = f(x) + c, \; x \in R$$

with

$$c = \int_R x^2 \mu(dx).$$

(It may be worth pointing out here that the counter example in the case of R_+ given above also serves as a counter example in the present case if R_+ is replaced by R.)

We shall now establish the following theorem answering the question of identification of the solution of (4.1) partially.

<u>Theorem</u> If f in (4.1) is not a function which is identically equal to a constant a.e.[L] on R_+ and f is either increasing a.e.[L] on R_+ or decreasing a.e.[L], then the equation cannot be valid unless either μ is a non-arithmetic measure and f is of the form

$$f(x) = \begin{cases} \gamma + \alpha(1 - e^{-\eta x}) & \text{a.e.[L] if } \eta \neq 0 \\ \gamma + \beta x & \text{a.e.[L], if } \eta = 0 \end{cases}$$

or μ is arithmetic with span λ for some λ, and f is of the form for which

$$f(x + n\lambda) = \begin{cases} f(x)e^{-n\lambda\eta} + a'(1 - e^{-n\lambda\eta}) & \text{a.e.[L] if } \eta \neq 0 \\ f(x) + \beta' n & \text{a.e.[L], if } \eta = 0 \end{cases}$$

where α, β, γ, α', β' are constants and η is such that

$$\int_{R_+} \exp(-\eta x)\mu(dx) = 1.$$

(From the statement of the theorem, it is implicit that if $\mu(R_+) = \infty$, then there is no solution to (4.1); this is also so if $\int_{R_+} x\mu(dx) = \infty$ when $\eta = 0$ and μ is non-arithmetic.)

<u>Proof</u> There is no loss of generality in assuming that f is increasing. Define for each $x \in R_+$

$$H_x(y) = f(x + y) - f(y), \; y \in R_+.$$

In view of the discussion in Remark 1, it follows that there is no loss of generality in taking f to be continuous. In that case, we get $H_x(\cdot)$ to be a continuous function on R_+ such that

$$\int_{R_+} H_x(y + z)\mu(dz) = H_x(y), \; y \in R_+.$$

From Lau-Rao theorem, we conclude that

$$H_x(y) = \xi_x(y)\exp(-\eta y), \; y \in R_+, \tag{4.4}$$

where ξ_x is such that $\xi_x(y + s) = \xi_x(y) \; \forall \; y \in R_+$ and every support point s of μ. In the case of non-arithmetic μ, (4.4) implies

$$\begin{aligned} f(x + ny) &= f(x + \overline{n-1}\; y) + \xi_y(0)\exp[-\eta(x + \overline{n-1}\; y)] \\ &= f(x) + \xi_y(0)\exp(-\eta x)\sum_{k=1}^{n}\exp[-\eta(k-1)y] \\ &= f(x) + \xi_{ny}\exp(-\eta x), \; x, y \in R_+, \; n \geq 1. \end{aligned} \tag{4.5}$$

It is easy to check that if (4.5) is valid, then $\xi_y(0) \propto [1 - \exp(-\eta y)]$, $y \in R_+$ if $\eta \neq 0$ and $\xi_y(0) \propto y$ if $\eta = 0$. Consequently, it follows that if μ is non-arithmetic we have for every $x \in R_+$

$$f(x) - f(0) = H_x(0) \propto \begin{cases} 1 - \exp(-\eta x) & \text{if } \eta \neq 0, \\ x & \text{if } \eta = 0. \end{cases} \tag{4.6}$$

In the case of arithmetic μ with span λ, we have directly from (4.4)

$$f(x + n\ ) - f(n\lambda) = \xi_x(\lambda)\exp(-\eta n\lambda)$$

$$= [f(x) - f(0)]\exp(-\eta n\lambda),\ n = 0,\ 1, \ldots, x \in R_+$$

and hence for $x \in R_+$ and $n = 0, 1, \ldots$ we have

$$(4.7) \qquad f(x + n\lambda) - f(x) = [f(n\lambda) - f(0)] + [f(x) - f(0)](e^{-n\lambda\eta} - 1)$$

$$= \begin{cases} (f(x) - f(0) + \xi^*)(e^{-n\lambda\eta} - 1) & \text{if } \eta \neq 0 \\ n[f(\lambda) - f(0)] & \text{if } \eta = 0 \end{cases}$$

where $\xi^* = [f(0) - f(\lambda)]/[1 - \exp(-\lambda\eta)]$.

The part assertion for the arithmetic case of μ is now obvious.

Remark 4 Isham et al (1975) considered a special case of the above theorem with the additional conditions that f is non-negative and right continuous with $f(0) = 0$, and μ is a probability measure. This special case was used in obtaining a certain characterization of the Poisson process and its discrete analogue.

Remark 5 If R_+ in (4.1) is replaced by R, then under the assumption that f is not a function that is equal to a constant a.e.[L] on R and f is either increasing or decreasing a.e.[L] on R, it follows that every solution f of the equation (4.1) can be expressed as a convex combination of functions f_1 and f_2 of the form arrived at in the theorem above with η replaced respectively by η_1

and η_2 satisfying the conditions

$$\int_R \exp(-\eta_i x)\mu(dx) = 1, \; i = 1, 2.$$

4.1 Dugue's problem

Rossberg (1972) and more recently in an unpublished article Wolinsta-Welez and Szynal (1984) have considered the problem of identifying characteristic functions ϕ_1 and ϕ_2 (of probability distributions on R) for which the following equation holds

$$(4.1.1) \qquad (1 - c)\phi_1(t) + c\phi_2(t) = \phi_1(t)\phi_2(t), \; t \in R$$

with $0 < c < 1$. This is indeed an extended version of the problem posed earlier by Dugue for $c = 1/2$.

Rossberg (1972) solved the problem when at least one of the ϕ_i's is non-arithmetic and Wolinsta-Welez and Szynal (1984) when both ϕ and ϕ_2 are arithmetic. In both these papers, there is an assumption that the left extremity of the distribution corresponding to ϕ_1 is non-negative and the right extremity of the distribution corresponding to ϕ_2 is non-positive. We shall now show that under the assumptions made by these authors, the problem of identifying the solutions to (4.1.1) reduces to a straight-forward application of LRS theorems.

Let F_1 and F_2 be the distribution functions corresponding to ϕ_1 and ϕ_2 respectively. Assume that $F_1(0-) = 1 - F_2(0) = 0$. It is then obvious that

(4.1.1) yields

$$(4.1.2) \qquad cF_2(-x) = \int_{R_+} F_2(-x - y)dF_1(y), \; x \in R_+ - \{0\}.$$

If F_1 is nonarithmetic, (4.1.2) implies, in view of Lau-Rao theorem, that $F_2(-x) \propto \exp\{-ax\}$ for $x > 0$ and some $a > 0$, and consequently (4.1.1) yields $F_2(-x) = \exp\{-ax\}$, $x \in R_+$ for some $a > 0$. (This follows since under the given assumptions, the fact that $F_2(-x) \propto \exp\{-ax\}$, $x \in R_+ - \{0\}$ when used in (4.1.1) gives the following equation relative to probability measures of $\{0\}$ on both sides

$$(1 - c)F_1(0) + c[F_2(0) - F_2(0-)] = F_1(0)[F_2(0) - F_2(0-)]$$

and hence $F_2(0) = F_2(0-)$. From this it follows that if (4.1.1) is valid, then under the assumption that at least one F_i is non-arithmetic (and hence without loss of generality that F_1 is non-arithmetic), we have

$$F_1 = 1 - e^{-bx}, \; x \in R_+, \; F_2(-x) = e^{-ax}, \; x \in R_+$$

with $a > 0$ and b such that $b = ac/(1 - c)$. (The converse of the assertion is obvious.) This is the result of Rossberg (1972) but for his apriori restriction that $F_1(0) = 0$. On the other hand if F_1 is arithmetic, in view of LRS theorems, (4.1.2) implies that

$$(4.1.3) \qquad \phi_2(t) = 1 - \alpha + \alpha \frac{(1 - \beta)\exp(-ibt)}{1 - \beta \exp(-ibt)}, \; -\infty < t < \infty$$

for some $b > 0$, $\alpha \in [0,1]$ and some $\beta \in [0,1)$ with an additional requirement that the corresponding characteristic function ϕ_1 satisfies

$$(4.1.4) \qquad \phi_1(t)\{(c - \alpha)e^{ibt} - c\beta + \alpha\} = c\{(1 - \alpha)e^{ibt} - \beta + \alpha\}, \ -\infty < t < \infty.$$

If X_1 is a random variable corresponding to ϕ_1, then from (4.1.4) we have for $n \geq 1$

$$(c - \alpha)P\{X_1 = nb\} = (c\beta - \alpha)P\{X_1 = (n + 1)b\}$$

implying that either $P\{X_1 = nb\} = 0$ for $n \geq 1$ or $\alpha \geq c$. Further (4.1.4) gives

$$(\alpha - c\beta)P\{X_1 = 0\} = c(\alpha - \beta)$$

which yields that $\alpha \geq \beta$ whenever $\alpha \geq c$. Thus, it follows that if (4.1.4) holds, then either $\phi_1(t) \equiv 1$, or for $\alpha \geq \max\{\beta, c\}$ and for some $b > 0$

$$(4.1.5) \qquad \phi_1(t) = c\,\frac{\alpha - \beta + (1 - \alpha)\exp(ibt)}{\alpha - c\beta - (\alpha - c)\exp(ibt)}, \ -\infty < t < \infty$$

which is clearly a characteristic function satisfying (4.1.4). Then it easily follows that if (4.1.1) is satisfied with at least one of the ϕ_i's as arithmetic and the extremity assumptions are satisfied, then either $\phi_1 \equiv 1$ and $\phi_2 \equiv 1$ or ϕ_1 and ϕ_2 are of the type given respectively by (4.1.5) and (4.1.3) for some $\beta \in [0,1)$ and $\alpha \in [\max\{\beta, c\}, 1]$. This is indeed the result arrived at by Wolinsta-Welez and Szynal using a different approach.

The following example illustrates that Rossberg-Wolinsta Welez-Szynal characterization of (ϕ_1, ϕ_2) satisfying (4.1.1) does not remain valid if the assumption that $F_1(0-) = 1 - F_2(0) = 0$ is dropped.

<u>Example</u> For a real $\theta \neq 0, 1$, let

$$\phi_1(t) = [(1 + it)(1 - \theta it)(1 - \frac{\theta it}{\theta - 1})]^{-1}, \ -\infty < t < \infty$$

and

$$\phi_2(t) = \phi_1(-t), \ -\infty < t < \infty.$$

Observe that (4.1.1) is satisfied with $c = 1/2$ and ϕ_1 and ϕ_2 are non-arithmetic. However, ϕ_1 and ϕ_2 are not of the form given by Rossberg.

In the above counter example, we have $F_1(0-) = 1 - F_2(0) > 0$. It may be noted that there also exist examples illustrating the point with either $F_1(0-) = 0$ or $1 - F_2(0) = 0$. In particular, if we take $c = \alpha/(1 - \alpha)$

$$\phi_1(t) = (1 - it)^{-2}, \ -\infty < t < \infty$$

$$\phi_2(t) = (1 + \beta\sqrt{\alpha}\, it)^{-1}(1 - \beta^{-1}\sqrt{\alpha}\, it)^{-1}, \ -\infty < t < \infty$$

with $\alpha = (\beta^2 - 1)^2/4\beta^2$ and $\beta > 1$, we have an example with $F_1(0-) = 0$. (The existence of an example with $1 - F_2(0) = 0$ follows by symmetry.)

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The problem of identifying solutions of general convolution equations relative to a group has been studied in two classical papers by Choquet and Deny (1960) and Deny (1961). Recently, Lau and Rao (1982) have considered the analogous problem relative to a certain semigroup of the real line, which extends the results of Marsaglia and Tubilla (1975) and a lemma of Shanbhag (1977). The extended versions of Deny's theorem contained in the papers by Lau and Rao, and Shanbhag (which the authors refer to as LRS theorems) yield as special cases improved versions of several characterizations of exponential, Weibull, stable, Pareto, geometric, Poisson and negative binomial distributions obtained by various authors during the last few years. In this paper the authors review some of the recent contributions to characterization of probability distributions (whose authors do not seem to be aware of LRS theorems or special cases existing earlier) and show how improved versions of these results follow as immediate corollaries to LRS theorems. The authors also give a short proof of Lau-Rao theorem based on Deny's theorem and thus establish a direct link between the results of Deny (1961) and those of Lau and Rao (1982). (CONTINUED)

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ITEM #18, SUBJECT TERMS, CONTINUED: Lau-Rao theorem; Shanbhag's lemma.

ITEM #19, ABSTRACT, CONTINUED: A variant of Lau-Rao theorem is proved and applied to some characterization problems.